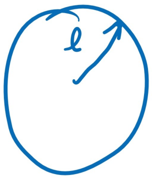
Lecture 6

Random walks in random environment in $d \geq 2$

Directional transience

For a given $\ell \in S^{d-1}$, $A_\ell = \{X_n \cdot \ell \to \infty\}$

$A_{-\ell} = \{X_n \cdot \ell \to -\infty\}$, $O_\ell = \left\{ \begin{array}{l} \limsup_{n\to\infty} X_n \cdot \ell = +\infty \\ \liminf_{n\to\infty} X_n \cdot \ell = -\infty \end{array} \right\}$



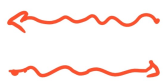
0-1 laws:

Kalikow's 0-1 law$^{\circledast}$: IID, elliptic, $\mathbb{P}^0(O_\ell) \in \{0,1\}$ equiv. $\mathbb{P}^0(A_\ell \cup A_{-\ell}) \in \{0,1\}$

$\circledast$ without unif. ellipticity Merkl - Zerner 2001

Major open question 1: P IID, unif. elliptic. Is $\mathbb{P}^0(A_\ell) \in \{0,1\}$?

Merkl - Zerner 2001: IID, elliptic, in $d=2$, $\mathbb{P}^0(A_\ell) \in \{0,1\}$


$(1, 1/2)$

$(-1, 0)$

Counterexample of Merkl-Zerner 2001:

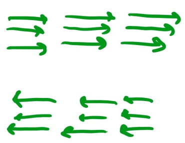
In $d=2$ (not essential here) there exists a stationary, ergodic and elliptic environment measure P under which:

$$\mathbb{P}^0(A_\ell) = \mathbb{P}^0(A_{-\ell}) = 1/2 \quad \text{for some } \ell. \ (\text{i.e., } \ell = \tfrac{1}{\sqrt{2}}(1,1))$$

Proof:

As a first step, construct P which is not even elliptic

Non-example: $P \sim$



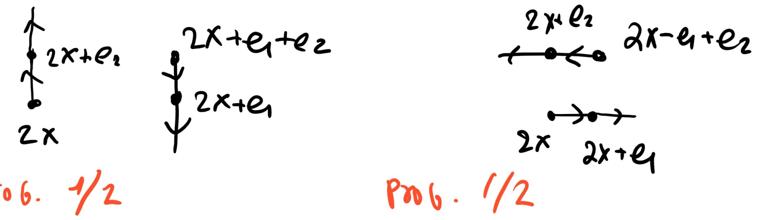
$\frac{1}{2}$

$\frac{1}{2}$

This choice of P is not ergodic

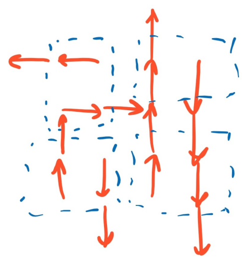
Also, first describe an env. measure $P_{(0,0)}$ which is inv. only to shifts in $2\mathbb{Z}^2$.

At each $x \in 2\mathbb{Z}^2$, put one of the following two distributions on the four vxs : x, $x+e_1$, $x+e_2$, $x+e_1+e_2$



Arrows mean a transition with prob. 1.

Put such an enviro. on 4 vxs at each vx of $2\mathbb{Z}^2$ independently



<u>properties</u>

stationary vx $2x \in 2\mathbb{Z}^2$

① Under $P^{2x}_{(0,0)}$, $\frac{1}{2} X_{2n} \cdot (1,1) = x \cdot (1,1) + n$ deterministicaly

$\frac{1}{2} X_{2n} \cdot (1,-1) =$ simple RW on $\mathbb{Z}$ starting at $x \cdot (1,-1)$

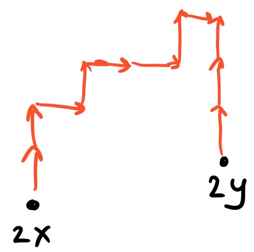
Consequently, $\frac{X_n}{n} \xrightarrow[n \to \infty]{} (\frac{1}{2}, \frac{1}{2})$ a.s.

② Under $P^{2x+e_1+e_2}_{(0,0)}$, $\frac{1}{2} X_{2n} \cdot (1,1) = x \cdot (1,1) - n$ deterministicaly

$\frac{1}{2} X_{2n} (1,-1) =$ simple RW on $\mathbb{Z}$ starting at $x \cdot (1,-1)$

$\Rightarrow \frac{X_n}{n} \xrightarrow[n \to \infty]{} (\frac{-1}{2}, \frac{-1}{2})$ a.s.

③ All paths starting at $2\mathbb{Z}^2$ eventually coalesce.



I.e, if $2x, 2y \in 2\mathbb{Z}^2$, their trajectories coalesce, a.s

This follows from property ①. Let one walk progress while the other waits until the projections of the walks on $(1,1)$ become equal.

Then, let both walks progress together until they coalesce.

This occurs eventually since one-dim. simple RW is recurrent.

(4) $P_{(0,0)}$ is $2\mathbb{Z}^2$-invariant and $2\mathbb{Z}^2$-ergodic ex.

(E.g. by Kolmogorov's 0-1 law and an approximation of inv. events by tail events)

To get a stationary and ergodic measure $\mathbb{P}$, do the following

$P_{(1,0)} = P$ translated by $(1,0)$

$P_{(0,1)} =$ " ———— " $(0,1)$

$P_{(1,1)} =$ " ———— " $(1,1)$

let $\underline{P} = \frac{1}{4}\left(P_{(0,0)} + P_{(0,1)} + P_{(1,0)} + P_{(1,1)}\right)$

It is clear that $\underline{P}$ is stationary (to all translations)

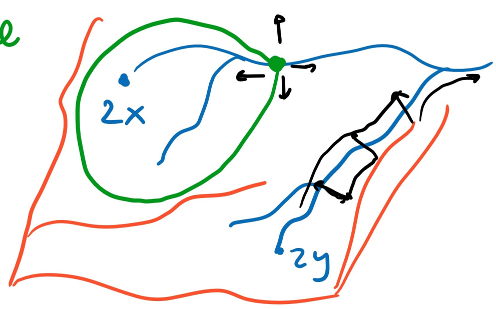
and it turns out that $\underline{P}$ is also ergodic ex.

This is thus a counterexample fulfilling the requiered properties except ellipticity

To get also ellipticity, we modify P

The previously-mentioned coalescence of Paths implies that the environment transitions form exactly two trees: A single tree with ancestors in the $(1,1)$ direction and its "dual" tree with ancestors in the $(-1,-1)$ direction.

height of the tree determines $E_{\omega} x$



Moreover in each tree, the subtree below every point is finite (as otherwise the infinite branches would split the plane and disrupt the coalescence of the other tree)

Define $\mathbb{P}'$ - a new environ. measure - as follows:

① First, sample ω from $\mathbb{P}$

② Create a new env. ω' by setting:

$$\omega'(x,e) = \begin{cases} 1-3\varepsilon_{\omega,x} & \omega(x,e)=1 \\ \varepsilon_{\omega,x} & \omega(x,e)=0 \end{cases}$$

where $\varepsilon_{\omega,x} = \frac{1}{(10+h(\omega,x))^2}$, and $h(\omega,x)$ = height of the tree below x

- Since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is summable, the walk will eventually stay on a single tree forever.
- The measure P' is a translation-covariant fcn. of $\mathbb{P}$. Hence, it is also stationary and ergodic. But now it is also elliptic.
- By symmetry, still $\mathbb{P}'^{0}(A_\ell) = \mathbb{P}'^{0}(A_{-\ell}) = \frac{1}{2}$ for $\ell = \frac{1}{\sqrt{2}}(1,1)$

The constructed environment $\mathbb{P}'$ is ergodic, but not mixing

Zeitouni notes that with considerable effort, in $d \geq 3$, there are stat. erg. examples which are unif. elliptic and polynomially mixing.

Beyond directional transience

Next question is the law of large numbers.

Drewitz Ramirez thm 2.7

Thm: (slight enhancement of Zerner 2002 and Zeitouni 2004)

Assume $\mathbb{P}$ IID and unif. elliptic, then there exist deterministic $\ell \in S^{d-1}$ and $v_+, v_- \in [0,1]$ s.t.

$$\frac{X_n}{n} \xrightarrow[n\to\infty]{} v_+ \mathbb{1}_{A_\ell}\, \ell + v_- \mathbb{1}_{A_{-\ell}} \cdot (-\ell) \qquad \mathbb{P}^0\text{-a.s.}$$

elliptic is not enough

Conj: In $d \geq 2$, under $\mathbb{P}$ IID and unif. elliptic

A_ℓ Directional transience in dir. ℓ $\Longrightarrow$ ballisticity in dir. ℓ — $\frac{X_n \cdot \ell}{n}$ has a pos. limit.

$X_n \cdot \ell \to \infty$

Additional Conj:

Conj: $\mathbb{P}$ IID, unif. elliptic (maybe even elliptic), then in $d \geq 3$ the walk is transient.

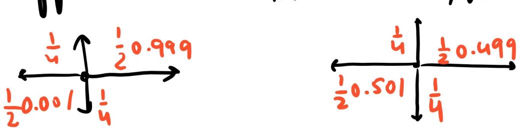
Conj: $\mathbb{P}$ IID, unif elliptic, will satisfy an annealed central limit theorem in $d \geq 3$ (Maybe also in $d=2$)

Lastly, Berger 2008 improved the law of large numbers in $d \geq 5$:
$\mathbb{P}$ IID, unif elliptic, $d \geq 5$ then there exists $\ell \in S^{d-1}$ and $v \in [0,1)$ s.t.

$$\frac{X_n}{n} \longrightarrow v \mathbb{1}_{A_\ell} \cdot \ell \quad \mathbb{P}^0\text{-a.s.}$$

Kalikow example: $d=2$

Suppose $\mathbb{P}$ is IID with two possible values for $\omega(0,\cdot)$



prob 0.999   prob. 0.001

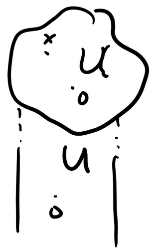
How to show that $\mathbb{P}^0(A_{(0,1)}) = 1$?

We seek conditions that are checkable (hopefully) which imply directional transience and more. Two such cond. are well known: Kalikow's cond. and Sznitman's (T) cond. (and later polynomial versions.)

Kalikow's condition

The underlying idea is to define a deterministic random walk with a drift and bound the original walk by the new walk.

In fact, we introduce many deterministic walks, one for each connected $U \subseteq \mathbb{Z}^d$, $0 \in U$, $U \neq \mathbb{Z}^d$



Roughly, the contribution of the drift at some $x \in U$ to the overall drift for the walk started at zero and stopped when it exits U is the number of visits to x times the drift at x.

For each U as above, define the transition prob. for a walk starting at 0 and stopped at U^c by

$x \in U$ $\quad e \in \{\pm e_i\}_{i=1}^d$, $\quad$ Prob. to go from x to $x+e$

$$\hat{P}_U(x,e) = \frac{E^0\left(\sum_{n=0}^{\tau_{U^c}} 1_{X_n = x} \cdot \omega(x, x+e)\right)}{\underbrace{E^0\left(\sum_{n=0}^{\tau_{U^c}} 1_{X_n = x}\right)}_{\text{expected number of visits to } x}}$$

$\tau_{U^c} = \min\{n \geq 0 : X_n \notin U\}$

Lemma: (Kalikow 1981)

Let $\mathcal{U}$ be as above. Assume $\hat{P}_{\mathcal{U}}(\tau_{\mathcal{U}^c} < \infty) = 1$

Then, $\forall v \notin \mathcal{U}$, $\hat{P}_{\mathcal{U}}(X_{\tau_{\mathcal{U}^c}} = v) = \mathbb{P}^0(X_{\tau_{\mathcal{U}^c}} = v)$

In particular, $\mathbb{P}^0(\tau_{\mathcal{U}^c} < \infty) = 1$

Proof:

Set $g_\omega(x) = E^0_\omega\left(\sum_{n=0}^{\tau_{\mathcal{U}^c}} \mathbb{1}_{X_n = x}\right)$ the expected number of visits to x in env. ω when starting at 0.

Note the usual recurrence, $g_\omega(y) = \mathbb{1}_{y=0} + \sum_{\substack{x \in \mathcal{U} \\ y \sim x}} \omega(x, y-x) g_\omega(x)$

Note also $\hat{P}_{\mathcal{U}}(x, y-x) = \dfrac{E^0(g_\omega(x)\omega(x, y-x))}{E^0(g_\omega(x))}$

Thus,

$$E^0(g_\omega(y)) = \mathbb{1}_{y=0} + \sum_{\substack{x \in \mathcal{U} \\ y \sim x}} \mathbb{E}^0(\omega(x, y-x) g_\omega(x)) = \mathbb{1}_{y=0} + \sum \hat{P}_{\mathcal{U}}(x, y-x) \mathbb{E}^0(g_\omega(x)) \quad (*)$$

In addition, set $\hat{\Pi}_n(y) = \hat{E}_{\mathcal{U}}\left(\sum_{j=0}^{\tau_{\mathcal{U}^c} \wedge n} \mathbb{1}_{X_j = y}\right)$ to be the expected number of visits to y, by time n, in the deterministic env.

Then, $\hat{\Pi}_0(y) = \mathbb{1}_{y=0}$ and $\hat{\Pi}_{n+1}(y) = \mathbb{1}_{y=0} + \sum_{\substack{x \in \mathcal{U} \\ x \sim y}} \hat{P}_{\mathcal{U}}(x, y-x) \hat{\Pi}_n(x) \quad (**)$

Consequently, by $(*)$ and $(**)$

$$\mathbb{E}^0(g_\omega(y) - \hat{\Pi}_{n+1}(y)) = \sum_{\substack{x \in \mathcal{U} \\ x \sim y}} \hat{P}_{\mathcal{U}}(x, y-x)\left(\mathbb{E}^0(g_\omega(x) - \hat{\Pi}_n(x))\right)$$

For any $y \in \mathcal{U} \cup \partial(\mathcal{U})$

By induction on n, $\mathbb{E}^0(g_\omega(y)) \geq \hat{\Pi}_n(y)$ $\forall n \geq 0$, $\forall y \in \mathcal{U} \cup \partial\mathcal{U}$

Hence also, $\mathbb{E}^0(g_\omega(y)) \geq \lim_{n \to \infty} \hat{\Pi}_n(y)$

In particular for $y \in \partial\mathcal{U}$ $\quad \mathbb{P}^0(X_{\tau_{\mathcal{U}^c}} = y) \geq \hat{P}_{\mathcal{U}}(X_{\tau_{\mathcal{U}^c}} = y)$

Summing over $y \in \partial U$,

$$\mathbb{P}^0(\tau_{U^c} < \infty) \geq \hat{P}_U(\tau_{U^c} < \infty) \overset{\text{by assumption}}{=} 1$$

$$\Longrightarrow \mathbb{P}^0(\tau_{U^c} < \infty) = 1 \quad \text{and} \quad \forall y \in \partial U, \; \mathbb{P}^0(X_{\tau_{U^c}} = y) = \hat{\mathbb{P}}_U(X_{\tau_{U^c}} = y)$$

Kalikow cond. in direction $\ell \in S^{d-1}$

$$\inf_{\substack{U \text{ as above} \\ x \in U}} \underbrace{\sum_{|e|=1} \hat{P}_U(x, x+e)\, \ell \cdot e}_{\text{expected drift in direction } \ell \text{ in the walk } \hat{\mathbb{P}}_U \text{ when at } x.} > 0$$

Thm (Kalikow, Sznitman-Zerner)

Under this cond. $\mathbb{P}^0(A_\ell) = 1$

and even $\frac{X_n \cdot \ell}{n} \xrightarrow[n \to \infty]{} v_\ell > 0$ $\quad \mathbb{P}^0$ a.s.